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# An arc-length continuation process for flutter solutions.

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**SUMMARY:**
The frequency domain modal analysis of the aeroelastic behavior of a structure subjected to gusty winds requires the determination of the eigenvalues of the aeroelastic system and the associated mode shapes for all subcritical wind speeds. These frequencies and modes are obtained at a chosen wind speed by solving the eigenvalue problem, which has to be iteratively solved because of the frequency dependence of the stiffness matrix. Classical solution schemes appear to be poorly efficient in this context as they rely on algorithms optimized for constant matrices and discard a consequent part of their outputs. This paper presents an alternative algorithm to perform the solution of the nonlinear generalized eigenvalue problem, under the form of an arc-length continuation process. The advantages offered by this method rely on the consideration of one mode at a time, which makes it more suitable for frequency dependent matrices as one single frequency at a time is involved in the system. Furthermore, it allows for a very systematic establishment of the modal basis by preventing any modal swapping between two sampled wind speeds.

*Keywords: pre-flutter behavior, numerical analysis, nonlinear continuation process.*

## 1. INTRODUCTION

Typical linear aeroelastic problems are of the form

$$\mathbf{M}_s\ddot{\mathbf{q}}(t)+\mathbf{C}_s\dot{\mathbf{q}}(t)+\mathbf{K}_s\mathbf{q}(t)-\frac{1}{2}\rho U^2\mathbf{Q}(k)\mathbf{q}(t)=0 \tag{1}$$

where $\mathbf{M}_s$, $\mathbf{C}_s$, $\mathbf{K}_s$ are the structural mass, damping and stiffness matrices, $\mathbf{Q}(k)$ is the complex generalized aerodynamic force matrix, $k=\omega b/U$ is the reduced frequency, $\omega$ is the frequency in rad/s, $b$ is a characteristic length and $\mathbf{q}(t)$ is the matrix of modal coordinates. Equation (1) is a time-frequency domain equation and several different approaches have been proposed for its eigensolution, such as the $p-k$ method (Rodden and Bellinger, 1982) or the $g$ method (Chen, 2000). As the airspeed varies, two or more eigenvalues can approach each other or even intersect, such that mode tracking can become problematic. Reduced frequency lining-up (Rodden, 1987) or predictor-corrector techniques have been applied to mitigate this problem but their effectiveness is strongly dependent on the airspeed increment used. Manual sorting of the eigenvalues is the last resort to ensure proper mode tracking. The method presented here offers a convenient and efficient alternative to prevent mode swapping.

## 2. DESCRIPTION OF THE METHOD

The proposed method consists in an arc-length continuation process to solve mode by mode the generalized eigenvalue problem. Focusing exclusively on a chosen mode $i$, it reads

$$-\lambda_i^2 \mathbf{M}(\omega_i, U)\phi_i + \mathrm{i}\lambda_i \mathbf{C}(\omega_i, U)\phi_i + \mathbf{K}(\omega_i, U)\phi_i = 0 \tag{2}$$

where $\mathbf{M}(\omega, U)$, $\mathbf{C}(\omega, U)$ and $\mathbf{K}(\omega, U)$ refer respectively to the matrices of mass, damping and stiffness. Each of these may be expressed as the sum of a constant and system related contribution and another variable part introducing the aeroelastic effects such that $\mathbf{M}(\omega, U) = \mathbf{M}_s + \mathbf{M}_{\text{ae}}(\omega, U)$, $\mathbf{C}(\omega, U) = \mathbf{C}_s + \mathbf{C}_{\text{ae}}(\omega, U)$ and $\mathbf{K}(\omega, U) = \mathbf{K}_s + \mathbf{K}_{\text{ae}}(\omega, U)$. For example, following the modified $p-k$ formulation (Chen, 2000),

$$\mathbf{M}_{\text{ae}}(\omega, U) = \mathbf{0},\ \mathbf{C}_{\text{ae}}(\omega, U) = -\frac{1}{2k}\rho U^2 \Im(\mathbf{Q}(k)),\ \mathbf{K}_{\text{ae}}(\omega, U) = -\frac{1}{2}\rho U^2 \Re(\mathbf{Q}(k))$$

If $n$ refers to the number of nodes, this problem stages $4n$ real unknowns for given air speed $U$, respectively $2n$ for the real and imaginary parts of $\lambda_i$, and $2n$ for the real and imaginary parts of the complex eigenvectors $\phi_i$. Therefore, this equation is supplemented by a normalization condition of the eigenvectors to close the system of equation

$$\Re\{\phi_i\} \cdot \Re\{\phi_i\} = 1 \quad \text{and} \quad \Im\{\phi_i\} \cdot \Im\{\phi_i\} = 1 \qquad i = 1, \ldots, n. \tag{3}$$

Equations (2) and (3) form a set of nonlinear algebraic equations and no longer an eigenvalue problem. It may be expressed under the form $\mathbf{f}(\mathbf{x}, U) = \mathbf{0}$, with $\mathbf{x}^T = \left[\lambda_i,\ \phi^T\right]^T$. Equation (3) with non-zero RHS prevents convergence to the trivial solution $\phi_i = \mathbf{0}$. Introducing $\mathscr{D}$, the $2n+3$ dimensional space defined by the reals unknowns, —namely air speed $U$, eigenvalues $\lambda_i$ and eigenvectors $\phi_i$—, and starting from a known point $p_0 \in \mathscr{D}$ solution of $\mathbf{f}(\mathbf{x}, U) = 0$, the arc length method consists in finding the intersection of the objective function $\mathbf{f}(\mathbf{x}, U)$ with the hypersphere of radius $r$, defined in $\mathscr{D}$ and centered on $p_0$. The situation is schematically illustrated in 2D in Figure 1. The equation of the hypersphere reads

$$r^2 = \left(\frac{U - U_0}{U_{\text{ref}}}\right)^2 + \left(\frac{\Re(\lambda) - \Re(\lambda_0)}{\Re(\lambda_{\text{ref}})}\right)^2 + \left(\frac{\Im(\lambda) - \Im(\lambda_0)}{\Im(\lambda_{\text{ref}})}\right)^2 + \sum_k \left[\Re(\phi_k) - \Re(\phi_{k,0})\right]^2 + \left[\Im(\phi_k) - \Im(\phi_{k,0})\right]^2. \tag{4}$$

where the quantities with index $_{\text{ref}}$ are scaling parameters, but can be taken equal to 1 and adapted if necessary. Linearizing $\mathbf{f}(\mathbf{x}, U)$ around a point $p_0 = (\mathbf{x}_0, U_0) \in \mathscr{D}$ with $x_0$, and $U_0$ such that $\mathbf{f}(\mathbf{x}_0, U_0) = \mathbf{0}$ and solving the so formed system for $\mathbf{x}$ gives

$$\mathbf{x} \approx \mathbf{x}_0 - \mathbf{J}_{\mathbf{x}}^{-1}(\mathbf{x}_0, U_0)\left[\mathbf{f}(\mathbf{x}_0, U_0) + \mathbf{J}_U(\mathbf{x}_0, U_0)(U - U_0)\right] \qquad p \sim p_0 \tag{5}$$

with $\mathbf{J}_{\mathbf{x}}(\mathbf{x}, U) = \dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}(\mathbf{x}, U)$ and $\mathbf{J}_U(\mathbf{x}, U) = \dfrac{\partial \mathbf{f}}{\partial U}(\mathbf{x}, U)$. Introducing (5) in the system formed by (2) and (3) provides a quadratic equation in $U$

$$(1 + \mathbf{b}^T\mathbf{S}^T\mathbf{S}\mathbf{b})U^2 - 2(U_0 - \mathbf{a}^T\mathbf{S}^T\mathbf{S}\mathbf{b})U + U_0^2 - (dR\, U_{\text{ref}})^2 + \mathbf{a}^T\mathbf{S}^T\mathbf{S}\mathbf{a} = 0 \tag{6}$$

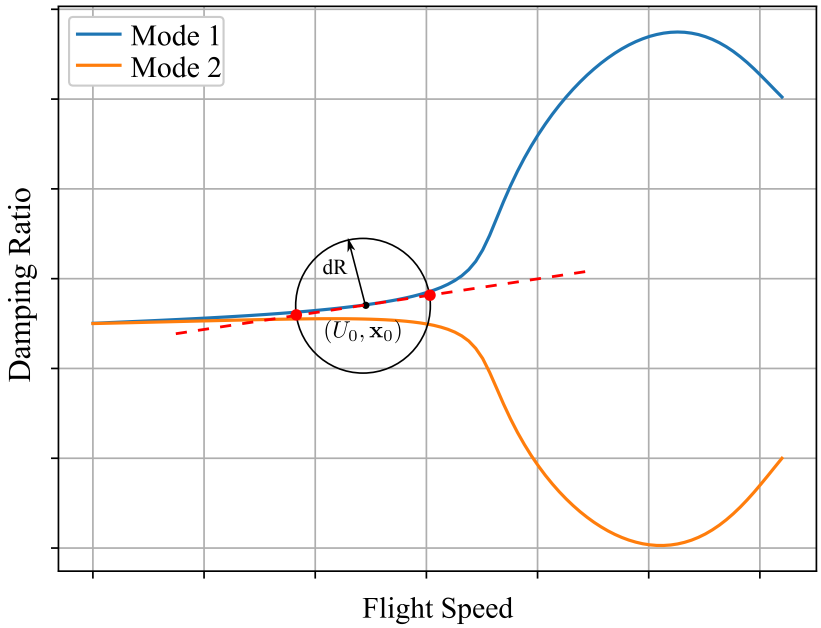

**Figure 1.** Illustration of the method in a 2D space.

with $\mathbf{a} = \mathbf{x} - \mathbf{x}_0 - \mathbf{J}_x^{-1}(\mathbf{x}, U)\left[\mathbf{f}(\mathbf{x}, U) - \mathbf{J}_U(\mathbf{x}, U)U\right]$, $\mathbf{b} = -\mathbf{J}_\mathbf{x}^{-1}(\mathbf{x}, U)\mathbf{J}_u(\mathbf{x}, U)$ and $\mathbf{S}$ is a diagonal rescaling matrix containing the reference coefficients $\Re(\lambda_{\text{ref}})$, $\Im(\lambda_{\text{ref}})$. This equation is easily solved analytically to obtain the two intersections with the sphere whose radius $r$ is fixed by the user. Among the two possible intersections, only the largest is of interest, as the process is initiated from wind-off conditions and progresses towards critical airspeed. At any time in the process, the next point is necessarily located at higher flight speed than the previous one. Rejecting systematically the lowest root prevents thus any fortuitous change in direction. The new flight speed $U$ is now used to evaluate a new $\mathbf{x}$ which is in turn used to get new $U$ *etc* until convergence is reached. Several iterations may be necessary to make sure that the intersection is close enough to the actual curve, depending on the tolerance fixed by the user. The discretization of the curve may be indirectly regulated by the choice of the sphere radius. This choice affects only the space between two consecutive points, but not the accuracy of the points themselves which depends exclusively on the fixed tolerance.

The presented method will be illustrated on two examples: a pitch/plunge model of a bridge deck and a multi-mode aeroelastic model of a plane wing.

## 3. CONCLUSION

The presented method solves the generalized eigenvalue problem for a chosen mode, and determines both the complex eigenvalues and the complex eigenvectors. As the eigenvalue problem is solved mode by mode, the mode swapping occurring when solving such systems with two modes having —even locally— very close eigenvalues is prevented. The semi-analytical solution for $U$ provides also a powerful control of the progress of the algorithm: no return back in the airspeed is possible. For this reason, the method offers a very systematic way to determine the variation of eigenfrequencies and damping ratios with airspeed without affecting the numerical performances that well-established methods are known to provide. Additional examples of applications and comparisons with existing techniques will be given in the full paper.

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